Evaluating Mathematical Knowledge Elements

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This paper is a brief description of a larger study conducted to examine mathematical responses from 90 first year university students. The main part of the study was concerned with identifying and describing the quality of existing mathematical knowledge that these students brought with them to university. The SOLO taxonomy (Biggs & Collis, 1982) technique was use d initially as the evaluation tool. However, certain limitations of the SOLO technique were identified which led to the development of a knowledge element coding technique. The focus of this paper is a brief description of this coding technique, its use and value in evaluating mathematical knowledge elements.

Several researchers, including Ball (1990, 1991), Eisenhart et al. (1993), Even (1993), Leinhardt (1988), Hiebert and Leferve (1988) and Gagne (1985) have generally agreed that mathematical knowledge has two components. The procedural component is that which consists of knowledge of algorithms, mathematical rules and procedures. The conceptual component is mathematical understanding. Other researchers, however, do not distinguish between the two components but define the growth of understanding mathematics as a continuing process of organising one's knowledge structures (Pirie & Kieren, 1994). Mathematical understanding from this perspective is defined as a whole dynamic process and not as a single or multi-valued acquisition, nor as a linear combination of knowledge categories. Regardless of whether mathematical knowledge is the acquisition of components or a growth process, a significant problem is its evaluation in order to determine whether or not the person has acquired it. Such an evaluation is particularly important in identifying potential secondary teachers of mathematics, since competency to teach mathematics requires proficiency in both mathematical knowledge and pedagogical knowledge (Ball, 1990, 1991; Eisenhart et al., 1993; Berliner et al., 1988).

Methodology

The sample consisted of 90 first-year university mathematics students. Fifty percent of the sample had received pre-tertiary education in Tasmania, forty percent from other Australian states and ten percent from overseas. These 90 students were enrolled in the pre-requisite mathematics unit for the Engineering, Science and Technology, and Mathematics and Science Education programmes. The students were classified into three groups according to their pre-tertiary mathematical background. Group A completed high level mathematics, group B the middle level and group C the low level mathematics. It was hypothesised that the quality of the students existing mathematical knowledge would be related to the differences in their pre-tertiary backgrounds. The data collection instrument was a questionnaire consisting of four mathematical items (see Appendix B). The instrument identified misconceptions relating to the product of two negative numbers, trigonometric and logarithmic functions and statistical variance. All the items required responses concerning processes in order to explain the procedures provided as cues.

To determine the types of mathematical knowledge elements that would facilitate and promote mathematical competence, a modified form of the SOLO taxonomy (Collis & Watson, 1991) task analysis technique was used for qualitative analysis. Several difficulties were encountered during the mapping of the students' responses using the SOLO technique (Collis & Watson, 1991). One of the main difficulties was categorising the responses into SOLO levels under the characteristics being specified for each level. For example, a uni-structural level was to be characterised by the presence of a single correct aspect of the specified mode. However, more than one type of 'single correct aspect' was observed during evaluation of the responses. This created classification and analysis problems. Therefore, the Knowledge Elements (KnE) task analysis technique was developed primarily to strengthen the validity of these SOLO level measures. The KnE is specifically for mathematical learned outcomes. It is a coding system to facilitate the categorisation of responses into complexity levels and to provide codes for descriptive measures. KnE takes into account procedural and conceptual mathematical knowledge. The codes are in the form of six broad bands of knowledge elements. Five bands of 'procedural' knowledge and the sixth band is for 'conceptual' knowledge elements. These bands are described in Table 1.

KnE Procedural Knowledge

The KnE procedural knowledge is based on Gagne's (1985) description of procedural knowledge as pattern-recognition and action-sequence. Pattern-recognition refers to the process of classifying information so that the person is able to recognise specific examples of concepts by relating them to general patterns. Action-sequence refers to the persons ability to carry out sequences of symbolic operations, that is, recognising patterns specified by the conditions whilst carrying out a sequence of actions. These actions are either covert, mental, actions or both covert and overt, physical, actions consisting of a 'series of steps in their correct sequence' (Gagne, 1985, p.103). Gagne suggested that these two forms of procedural knowledge are linked in performance. For example, consider the Trigonometric item: solve for x in Cos(2x+1)=0. Here, the pattern-recognition is knowing that this 'given cue' is not in the same form as similar cues such as y(2x+1) that can be expanded to 2xy + y. The action-sequence type knowledge for the 'given cue' follows on from pattern-recognition. That is, after recognising, covertly, that Cos(2x+1)is a trigonometric function, the person could proceed to the next action of giving meaning to the symbols '(2x+1)'. Knowing that (2x+1) represents an angle measurement corresponding to the Cosine value of zero, would lead to the next action of recalling what the angle size is. Once this is retrieved, the computation procedures would follow resulting in a solution for x. Note, however, that pattern-recognition can occur simultaneously during the process of action-sequence. For instance, knowing what angle size to recall requires the ability to recognise, compare and to eliminate angle sizes of other related trigonometric functions, e.g. sine and tangent. Anderson (1981) and Gagne (1985) have maintained that pattern-recognition and action-sequence procedures are closely connected and that the relationship between them is similar to that between concepts and rules. In addition, mastery of pattern-recognition is through generalisation and discrimination processes. Generalisation explains how a person can classify concepts they have never seen before. Therefore, generalisation increases the range of situations to which a procedure applies. For example, an individual who learned that km + 2m is the same as m(k+2) could generalise this pattern incorrectly to others of similar form as in Cos(2x+1) and log(x-1), seeing them, respectively, as Cos2x + Cos1 and logx - log1.

Discrimination, on the other hand, restricts the range of conditions to which a procedure can be applied. The statistic, trigonometric and logarithmic items are examples of situations with restricted conditions, see Appendix B. For instance, the evaluation of log(2x+1)=0 is restricted to procedures that meet the conditions for this situation only. Other procedures would be required for the evaluation of, say, log 2x=1.

Computation is a function or a dependent process of procedural knowledge. Computation involves speed and accuracy. These two components are dependent on the pattern-recognition processes of generalisation and discrimination. Computation is also part of the action-sequence procedures and in some cases it is the link between pattern-recognition and action-sequence. For example, if a person was asked; find the area of the largest circle inside a square of 2cm length, then the procedure that connects pattern-recognition or discriminating between shapes and area, to action-sequence, is computation of the required area. Accuracy and the rate in which the computation is carried out are essential features of this element of procedural knowledge. According to Collis (1990, p.5), computation in mathematics illustrates mastery of 'the link between the symbol systems themselves and the world'.

Procedural knowledge, therefore, involves more than knowing and recognising symbols, patterns, rules and algorithms. It also involves processes of generalisation, discrimination and computation, all of which are based on understanding and prior knowledge. Eisenhart et al. (1993, p.9) have defined procedural knowledge as 'mastery of computational skills and knowledge procedures for identifying mathematical components, algorithms, and definitions'. These authors have further described procedural knowledge of mathematics as consisting of two parts: (a) knowledge of format and syntax of the symbol representation system and (b) knowledge of rules and algorithms.

KnE Conceptual Knowledge

Knowledge comprising facts has been described by Anderson (1980, p. 222) as 'declarative knowledge'. According to Collis (1990), learning in the concrete symbolic mode leads to declarative knowledge which is demonstrated by 'an ability to make symbolic descriptions of the experienced world' (p.5). Declarative knowledge is knowledge that is expressed through 'the medium of a symbol system in a way that is publicly understandable' (Biggs & Collis, 1991, p.59). From these perspectives it seems that mathematics, a symbol system, is 'purely' declarative knowledge. Such perspectives also imply a necessity for understanding the mathematical concepts represented by the symbols. In other words, the link or relationship between a mathematical concept and its symbolic representation is understanding. It follows that one way to determine an individual's mathematical understanding is to obtain a measure of the individual's declarative knowledge. Declarative knowledge in this sense encapsulates both procedural and conceptual knowledge.

Conceptual knowledge in mathematics is defined as fundamental knowledge which underlies the structures for mathematical competence (Gagne, 1985). These structures are the relationships and interconnections of ideas that explain and give meaning to mathematical procedures represented by symbols (Eisenhart et al. 1993). From Gagne's (1985) perspective, conceptual knowledge is understanding in the form of knowledge organisation or sets of propositions, sets of recognised patterns, or some mixture of these. In the KnE framework, conceptual knowledge is demonstrated by a person's ability to make the relationships between declarative knowledge, symbols and concepts (Anderson, 1980; Collis, 1990; Biggs & Collis, 1991), and knowledge organisation or sets of pattern-recognition (Gagne, 1985). Thus, the sixth broad band of knowledge elements is called 'relationships'.

Table 1: K1 to K5 are the broad bands for procedural knowledge elements and K6 is the band for conceptual knowledge

Code	Knowledge Band	Knowledge description
K1	Recognition & Classification	Knowledge of what the situation is or the ability to classify the situation into categories of known or familiar knowledge. [Generalisation & Discrimination]
K2	Routine	Knowledge of the problem in the most routine sense. For example, substitution of values into an equation; knowing the 'operation' for the terms such as a product, to increase, and knowing when to expand an expression or to factorise. [Generalisation]
K3	Rules & Algorithms	Knowledge of rules and algorithms to compute the task. [Discrimination]
K4	Format & Syntax	Knowledge of format and syntax of symbol representation. [Discrimination]
K5	Computation	Knowledge of how to solve or compute correctly. [Computation]
K6	Relationships	Knowledge of relationships and interconnections of ideas that explain and give meaning to mathematical procedures.

The processes stated in the 'brackets' are the main processes involved in generating the particular knowledge element.

Scale Values For The KnE Descriptors

Since the KnE method was essentially to provide codes for elements of procedural and conceptual knowledge. These codes, assumed to be associated with the SOLO levels: pre-structural, uni-structural, multi-structural, relational, and extended abstract, provided new descriptions for the SOLO. For example, instead of perceiving a uni-structural as representing a single aspect of a mode, the new description is represented by several KnE codes. Thus, each SOLO level can be perceived as having quantifiable elements. A justification of this approach is provided by Cryer and Miller (1991, p.108) who suggested that for categorical measurement, classification of elements according to a common attribute, new variables can be created through arithmetic or functional operations on existing variables and such creations would constitute a change in the measure of the magnitude of the new variable. Since the KnE descriptors are essentially new variables being created from existing SOLO classifications, a set of arbitrary numerical values was employed to indicate the change in the magnitude of the new variables. That is, pre-structural=1 (PRE), uni-structural=2 (UNI), multi-structural=3 (MULT), relational=4 (REL), and extended abstract=5. In addition, because the SOLO levels were assumed to have no clearly defined boundaries between the upper and lower levels, the values of +0.5 and -0.5 were also introduced. A value of +0.5 was added to represent response structures that were border-line towards an upper category and -0.5 to represent structures border-line towards a lower category. This addition of border-line values was also to reflect the continuous or 'growth' nature of knowledge. Thus, the numerical values associated with the KnE codes were; No-Attempt = 0, PRE- = 0.5, PRE = 1.0, UNI- = 1.5, UNI = 2.0, UNI+ or MULT- = 2.5, MULT = 3, MULT+ = 3.5, REL = 4.0, REL+ = 4.5, and Extended Abstract = 5.0.

KnE Descriptors

The KnE descriptors described here refer to the categories obtained from the re-classification of the study data using the KnE task analysis method. To distinguish between the SOLO levels and those described by the KnE technique, the term 'descriptor' rather than 'level' was adopted. The KnE descriptors or categories are the same as the SOLO taxonomy, these being pre-structural (PRE), uni-structural (UNI), multi-structural (MULT), and relational (REL). However, the criterion for classification was different. That was, the KnE descriptors were classified in terms of knowledge elements rather than 'relevant aspects' of the mode of intellectual functioning as assumed in the SOLO taxonomy (Biggs & Collis, 1982). The responses to the four items: NEG, TRIG, LOG, and STAT, were recoded as K1 - Recognition & Classification, K2 - Routine, K3 - Rules & algorithms, K4 - Format & Syntax, K5 - Computation, and K6 - Relationships. To differentiate between correct and incorrect use of the particular knowledge element, a subscript x was used. For example, K2x would mean that incorrect routine knowledge was demonstrated. Descriptions of each KnE descriptor are as follows as well as a sample coding of a multi-structural SOLO task analysis map.

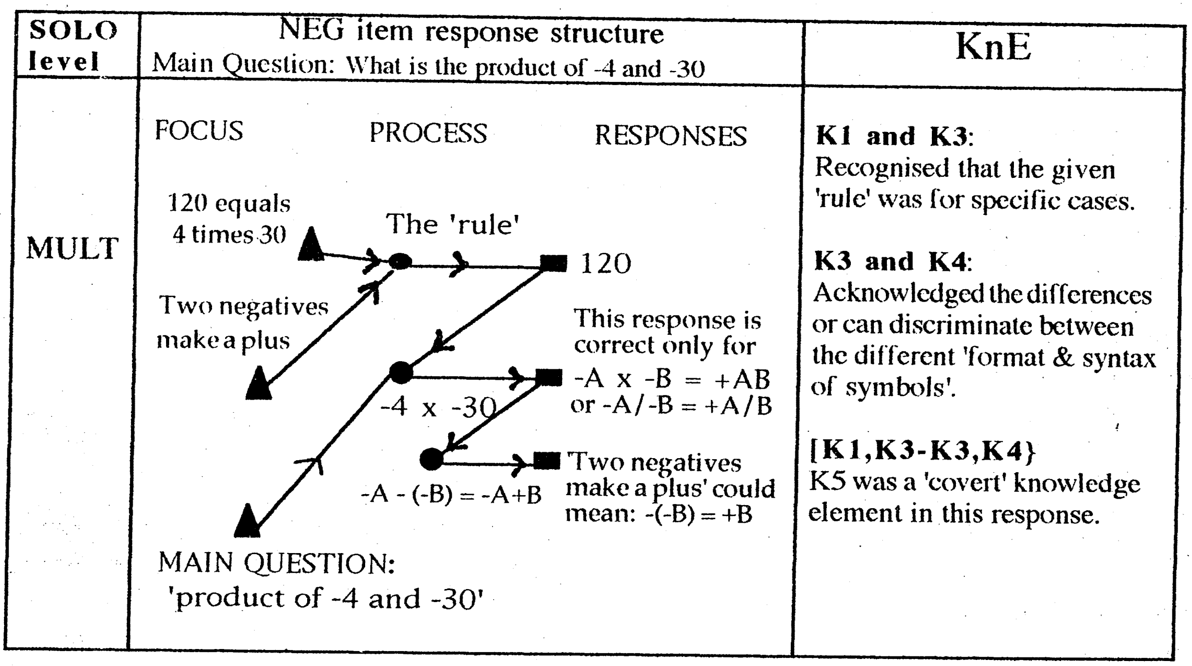
A pre-structural response (PRE) was characterised by failure to make correct generalisation and to carry out appropriate sequence of actions for the task. In other words, such a response reflected a lack of mastery of generalisation and discrimination processes which are required processes of pattern-recognition procedures.

A uni-structural response (UNI) was characterised by a sequence of K1, K2, K3, and K4 type knowledge elements. That is, K1 is an indication that the correct pattern-recognition was made for classifying the 'given cues' into known categories; followed by a covert or overt routine action (K2) while simultaneously recalling correct rules and/or algorithms (K3) and applying correct format and meaning to symbols (K4). A UNI+

response, on the other hand, would require an additional recognition of elements of the 'given cues' that could lead to K5 or computation.

A multi-structural response (MULT) was characterised by the presence of K5 or computation being correctly carried out. A MULT+ response was seen in attempts to utilise as much of the 'given cues' as appropriate to provide a 'justification' for the sequence of actions rather than for conceptual reasons. A sample coding of a multi-structural response is given below in Figure 1. The second column of Figure 1 is a task analysis using the SOLO mapping technique (Collis & Watson, 1991).

Figure 1: Sample coding of a multi-structural response using the KnE method



A relational response (REL) was characterised by evidence of 'justifications' for conceptual reasons. This process was classified as a K6 element, because it indicated an interconnection of the 'given cues' with the individual's conceptual understanding of the situation. It also provided an indication of mastery of the pattern-recognition and action-sequence processes. This process also appears to govern the action-sequence and the closure of the response. The REL+ response was characterised by references being made to a general or abstract form of the concept not represented in the 'given cues'.

A REL is distinguished from a MULT by the inclusion of K6. K6 could occur twice in REL responses, once in the beginning and once at the conclusion, usually as a statement to link the action-sequence as a whole unit of response. K6 acted to elaborate and organise knowledge. According to Gagne (1985, p.100), elaboration 'is the process of adding related knowledge to the new... [or given cues] ... knowledge' and organisation 'is the process of putting declarative knowledge into subsets and indicating the relationships among subsets'. She added that organisation enhances management of the 'limited-capacity of the working memory during retrieval' (p. 101).

To summarise the above descriptions, three distinct qualities of knowledge were found. These were: pre-mastery of pattern-recognition, concrete symbolisation and conceptual symbolisation. The pre-mastery of pattern-recognition type knowledge included responses classified as PRE-, PRE, and UNI-. The concrete symbolisation type knowledge comprised responses classified as UNI, UNI+, and MULT. The conceptual symbolisation type knowledge included responses classified as MULT+, REL, and REL+. There is a summary of the knowledge descriptors and their descriptions in Appendix A.

Results and Conclusion

The use of the KnE method in conjunction with the SOLO levels provided a clearer description of the quality of mathematical knowledge that constitute mathematical competence in the response structures . The SOLO technique alone was not sufficient in determining mathematical competence since SOLO is essentially for evaluating shifts in the cycle of learning within a developmental stage or mode (Biggs & Collis, 1991). Although mathematical competence has been described in SOLO terms as 'understanding in the concrete symbolic mode' (Collis & Romberg, 1991, p.96), it still begs the question of what exactly this understanding consists of. The KnE technique described above has been effectively used in describing mathematical competence for non-research mathematical items (Gates et al., 1995) as well as providing a mechanism for quantifying qualitative data.

It was hypothesised that the quality of the students' existing mathematical knowledge would be related to the differences in their pre-tertiary backgrounds. However, a Chi-square goodness-of-fit test was not significant ($p<0.01$) and it was concluded that there was no significant difference in the quality structure of responses from the three groups. Further analysis using a model to describe the interconnection of procedural and conceptual knowledge showed most of the responses from the three groups to be within the pre-mastery of pattern-recognition and concrete symbols, see Appendix A. The results strongly suggest that these first year university students enter university studies with mainly procedural knowledge of mathematics. Such findings could not have been achieved with the use of the SOLO taxonomy technique alone. The KnE technique has potential for evaluation and assessment of mathematical cognitive processes. Particularly evaluations and assessments of cognitive structures, knowledge elements, pertaining to mathematical competence in both the pre-tertiary and tertiary levels. This paper only provides a brief description of the KnE technique and a small part of a larger study. For a detail description of the research and methodology, the reader is encouraged to read the whole thesis which will be made available in the library at the University of Tasmania.

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APPENDIX A:

A summary of KnE descriptions

KnE Descriptors	Mathematical knowledge element description	KnE codes	Knowledge description
PRE⁻	Predominantly incorrect pattern-recognition procedures presumably because of lack of knowledge - DONT KNOW, or inability to recall learning - CANT REMEMBER or CANT RECALL.	$K1_x$, $K2_x$, $K3_x$	PRE-MASTERY OF PATTERN-RECOGNITION
PRE	Failure to make correct generalisation and discrimination as well as providing inappropriate sequence of actions for the task.	$K1_x$, $K2_x$, $K3_x$, $K4_x$, $K5_x$ [any combination of these]	
UNI⁻	Showed the ability to make a correct pattern-recognition. This may be followed by a routine action or a statement of an incorrect rule or algorithm.	K1, K2 or K1, K2, $K3_x$,K1, K3, $K4_x$	
UNI	Showed the ability to make correct pattern - recognition (K1); followed by routine action (K2). Demonstrated the ability to discriminate rules and algorithms(K3) and simultaneously apply correct format and meaning to symbols (K4).	K1, K2, K3, K4	CONCRETE SYMBOLISATION
UNI⁺	Showed all the elements as in UNI with an additional indication of procedures that could lead to correct computation -->K5	K1, K2, K3, K4 --> K5	
MULT	Showed all the elements as in UNI⁺ with the additional computation (K5) element.	K1, K2, K3, K4, K5	
MULT⁺	Showed all the elements as in MULT plus a justification for the preceding sequence of actions. A positive sign of a K6 element.	K1, K2, K3, K4, K5--> K6	CONCEPTUAL SYMBOLISATION
REL	Provided evidence of conceptual links between the procedural knowledge elements. Showed mastery of the processes involved in pattern-recognition and action-sequence.	[K6], K1, K2, K3, K4, K5, K6	
REL⁺	In addition to the elements as in REL, evidence of reference being made to a general form, abstract concept, of the concept not represented in the given cues.	[K6], K1, K2, K3, K4, K5, K6⁺	

APPENDIX B:

The Questionnaire Items

1. Statistics

Ten items were measured and four results were produced:
A class of year 11/12 students was given the information below to find the requested measures:

$$\sum_{i=1}^{10} x_i = 40 \qquad \left(\sum_{i=1}^{10} x_i\right)^2 = 1600$$

$$\sum_{i=1}^{10} x_i^2 = 194 \qquad \sum_{i=1}^{9} \left(x_i - \bar{x}\right)^2 = 24$$

Use these results to find the *mean*, $\bar{X}$, and *variance*, S_x^2, for these 10 items.

The class responses:

Students produced 3 different values for the *variance*, S_x^2:
i) 2.4 (ii) 3.4 (iii) 2.7

Q. Which *variance*, S_x^2, is the correct one? Please explain.

2. Logarithmic function

Simplify and evaluate for x,
$\log_{10}(2x+1) = \log_{10}(x-1)$

A student responded:

log (2x+1) = log (x-1)

log 2x + log 1 = log x - log 1 (log 1=0)
log 2x -log x = 0, ($\log_{10} x = 0$)

$x = 10^0$ ∴ x = 1

Q1. Is the student's response correct?
Q2. Please explain and show why you answered yes/no to Q1.

3. Negative Number

A high school student was asked the following question:
What is the product of -4 and -30?
A student responded:

"120 equals 4 times 30
Two negatives make a plus"

Q1. Would you say this student understands how to explain the products of negative numbers?
Q2. How would you explain the result of 120?

4. Trigonometric Function

Evaluate for x, Cos (2x+1) = 0
A Student responded:

Cos (2x + 1) = 0
Cos 2x + Cos 1 = 0
Cos 2x = -Cos 1
2x = -1
∴ x = -1/2

Q1. Is the student response correct?
Q2. Please explain and show why you answered yes/no to Q1.